Regional Mathematics (2001 to 2013)

RMO - 2001

1. Suppose $n > 2$ is an integer and $x_1, x_2, \cdots, x_n$ are real numbers such that $x_1 > x_2 > \cdots > x_n$. Prove that

$$\frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_3} + \cdots + \frac{1}{x_{n-1} - x_n} + \frac{1}{x_n - x_1} > 0.$$

2. If r_1, r_2, r_3, r_4, r_5 are the roots of the equation $x^5 + x^2 + 1 = 0$, and $p(x) = x^2 - 2$, find the value of the product

$$p(r_1)p(r_2)p(r_3)p(r_4)p(r_5).$$

3. Find all the real solutions of the system

$$x^2 + y^2 + \frac{2xy}{x+y} = 1, \quad \sqrt{x+y} = x^2 - y.$$

4. Let $\sigma(n)$ be the sum of all positive divisors of a positive integer n (including 1 and n) and $\phi(n)$ the number of positive integers less than n and prime to n.
Prove that $\sigma(n) + \phi(n) \geq 2n$.

5. Let $n > 1$ be an integer. Find the number of all permutations $(a_1, \cdots, a_n)$ of $(1, 2, \cdots, n)$ such that $a_i > a_{i+1}$ for exactly one index $i, 1 \leq i \leq n-1$.

6. The trapezium $ABCD$ is given with $AB \| DC$ and $AB = AC + CD$. E is the midpoint of $BD, BF \| CE$ and F lies on the line AC. Prove that
(i) AE and DF are perpendicular or BF.
(ii) C is the incentre of $\triangle DEF$ if and only if DA is perpendicular to AB,
(iii) $EF \| AD$ if and only if $AB = 3CD$.

7. ABC is a scalene triangle with circumcentre O and orthocentre H. A', B', C' are the reflections of the vertices A, B, C in the corresponding sides BC, CA and AB. If BC' and $B'C$ meet at D, then show that
(i) $OBDC$ is a cyclic quadrilateral and
(ii) DA' is parallel to the Euler line OH.

8. Suppose a, b are integers such that $a > b \geq 1$ and $ab(a+b)$ is a multiple of $a^2 + ab + b^2$.
Prove that $a - b > \sqrt[3]{3ab}$.

9. Given a party in which any two persons have exactly one friend in common, prove that there must be a 'host' who is everybody's friend.

RMO - 2002

1. $ABCD$ is a trapezium. The diagonals AC and BD intersect at M. The line through M parallel to AB meets AD at E and BC at F. Prove that EF is the harmonic mean of AB and CD, that is $\frac{1}{AB} + \frac{1}{CD} = \frac{2}{EF}$.
2. Let T_1 be a triangle. Let T_2 be the triangle whose sides have length equal to the length of the medians of T_1. Let T_3 be the triangle whose sides have length equal to the length of the medians of T_2. Show that T_3 is similar to T_1.
3. Find the remainder when the polynomial $x^{401}+x^{301}+x^{201}+x^{101}+x$ is divided by $x^4 - x$.
4. Let $a_1, a_2, \cdots, a_n$ be real numbers such that each $a_i \geq 1$. Show that $2^{n-1}(1 + a_1a_2 \cdots a_n) \geq (1 + a_1)(1 + a_2) \cdots (1 + a_n)$.
5. Consider a rectangle of size 8×10. It is divided into 80 unit squares by drawing lines parallel to its sides. Consider L-shaped figures formed by exactly four unit squares. One of the various possible figures is shown below. Find the total number of figures in the given rectangle which are congruent to the figure shown.
6. Let S be the set of all ordered $n-$tuples $(a_1, a_2, \cdots, a_n)$ such that
(i) each a_i is a natural number, (ii) $a_1 = 1$ and
(iii) $(a_{i+1} - a_i)$ equals 0 or 1 for $1 \leq i < n$.
7. Let N be the largest integer such that the decimal expansions of both N and $7N$ have exactly 2002 digits each. Find the 1001th digit, from the left, in N.
8. Let $a_1 = 2$. For $n \geq 2$, let a_n be the largest prime factor among all the prime factors of the following number:
$(1 + a_1a_2 \cdots a_{n-1})$. Prove that a_n can never be 5.

RMO - 2003

1. O is a point inside $\triangle ABC$. The lines joining the three vertices A, B, C to O cut the opposite sides in K, L, M respectively. A line through M parallel to KL cuts the line BC at V and AK at W. Prove that $VM = MW$.
2. Find all primes p such that $5^p + 4p^4$ is a perfect square.
3. Let $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$ be any sequence of positive odd integers. Prove that for every n, there is an integer k such that
$a_1 + a_2 + \cdots + a_n \leq k^2 \leq a_1 + a_2 + \cdots + a_n + a_{n+1}$.
4. Let a, b, c, d, e, f be real numbers such that all the roots of the polynomial
$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$
are real and positive. Determine all possible values of f.

5. Suppose AB and EF are non-intersecting chords in a circle, and that Q is a variable point on the arc AB remote from E and F. If the lines QE and QF cut AB into three segments of lengths x, y and z (in order), then prove that xz/y is independent of Q.
6. In an aeroplane, there are 63 rows with 6 seats in each row. The seats in each row are labelled 1 to 6 from left to right. Suppose 193 passengers are seated in the aeroplane. Show that there are at least two rows in which all the occupied seats have the same labels.

RMO - 2004

1. Determine all natural numbers n such that $\frac{n}{1!} + \frac{n}{2!} + ... + \frac{n}{n!}$ is an integer.
2. If $x, y, z > 0$ show that $\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} + \sqrt{\frac{z+x}{x+y+z}} \leq \sqrt{6}$
3. Let $n \in N$. Let a_n equal to the sum of the squares of the digits in the decimal representation of n. For example $a_{547} = 5^2 + 4^2 + 7^2 = 90$ Find the sum of $a_1 + ... + a_1 000$.
4. Let ABC be a triangle and D, E, F be the midpoints of BC, CA and AB respectively. Suppose G is the centroid of the triangle ABC. If $BDGF$ is a cyclic quadrilateral and $2BE = \sqrt{3}AB$, then prove that ABC is an equilateral triangle.
5. Let $f(x) = x^4 - x^3 + 8ax^2 - ax + a^2$ and $g(y) = y^2 - y + 6a$
(a) Prove that $f(x) = (x^2 - y_1 x + a)(x^2 - y_2 x + a)$, where y_1 and y_2 are the roots of the equation $g(x) = 0$
(b) Find all values of a such that the equation $f(x) = 0$ has four distinct real positive roots.
6. A pair of natural numbers (a, p) is said to be good if p is an odd prime and 3 is the smallest positive value of n such that p divides $a^n - 1$. If (a, p) is a good pair then find the smallest value of m such that $(a + 1)^m - 1$ is divisible by p.
7. A quadrilateral $ABCD$ has an inscribed circle. The inscribed circle touches AB and CD at E and F respectively. Prove that $\frac{AE}{EB} = \frac{DF}{FC}$ if and only if $ABCD$ is a cyclic quadrilateral.
8. Let $ABCD$ be an 11x11 square grid consisting of 121 unit squares and $PQRS$ be 5x5 square inside $ABCD$ as shown above (shaded region). Find the number of rectangles in the grid such that intersection of each rectangle with square $PQRS$ has area zero.

RMO - 2005

1. For each positive integer n, let $T_n = n(n+1)/2$. Find all pairs (n, m) of positive integers such that $n > m$ and $T_n - T_m = 2^k$ for some positive integer k.
2. Find all values of a for which the equation $x^3 + x^2 - x + a = 0$ has three integer roots.

3. A permutation $P = a_1a_2...a_n$ of the set $\{1, 2, ..., n\}$ is called *special* if there is exactly one j such that $1 \leq j \leq n-1$ and $a_j > a_{j+1}$. For example ,with $n = 5$, the permutations 13245,51234,24135 are special. Prove that the number of special permutations of the set $\{1, 2, ..., n\}$ is $2^n - 1 - n$.
4. Find all positive integers a, b such that the numbers $\frac{a^2+b}{b^2-a}$ and $\frac{b^2+a}{a^2-b}$ are both integers.
5. In convex quadrilateral $ABCD$, $\angle ABC$ is obtuse and $\angle CAB = \angle DBC$. Also the sides BC, AD and diagonal AC are of lengths which satisfy $BC^2 + AD^2 = AC^2$. Prove that $\angle ADB = \angle DCA$.
6. For each number in the set $\{n+1, n+2, ..., 2n\}$, consider its largest odd divisor and add all such largest odd divisors. Prove that the sum so obtained is n^2.
7. If x, y, z are positive numbers such that $x + y + z = 1$,
prove that

$$\frac{1+x}{1-x}\,\frac{1+y}{1-y}\,\frac{1+z}{1-z} \geq 8$$

8. On sides of AC and BC of an acute angled triangle ABC, rectangles $ACPQ$ and $BKLC$ are constructed outwardly. Assuming that these rectangles have equal areas, prove that the vertex C, the circumcenter O of ΔABC, and the midpoint M of segment PL are collinear.

RMO - 2006

1. Let x, y, z be positive real numbers such that $xyz = a$. If $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq x + y + z$, prove that $\frac{1}{x^k} + \frac{1}{y^k} + \frac{1}{z^k} \geq x^k + y^k + z^k$, for every positive integer k.
2. Find all integers m such that $m + 3$ and $m^2 + 3m + 3$ are perfect cubes.
3. Each year 8 subjects are taught by 4 teachers in a school. Every teacher teaches two subjects. At the end of this year they will meet to decide the course allotment for the next year. Find the number of ways in which the course distribution can be done so that each teacher teaches two courses and each teacher teaches at least one subject different from the subjects which he taught this year.
4. Let C be a point on the circle with center O and radius r. Chord AB of length r is parallel to radius OC. Let the line AO cut the circle in E and the tangent to the circle at C in F. If the chord BE cuts OC in L and if AL cuts CF in M, find the ratio $\frac{CF}{CM}$.
5. In the set of complex numbers solve te system of equations

$$x(x-y)(x-z) = 3$$
$$y(y-x)(y-z) = 3$$
$$z(z-x)(z-y) = 3.$$

6. An 8X8 board is divided into unit squares. Each unit square is painted red or blue. Find the numbet of ways of doing this so that each row and each column has odd number of blue squares.
7. Find all natural numbers x, y, z such that

$$\sqrt{\frac{2006}{x+y}} + \sqrt{\frac{2006}{y+z}} + \sqrt{\frac{2006}{z+x}}$$

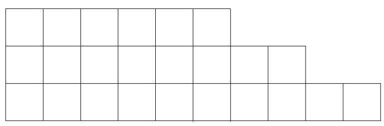
is a number.
8. Consider circle with center O and radius OA. Let C be a point on radius OA. Let P be a variable point on the circle. Join P and C. Q is a point on the circle such that P and Q are on the same side of line OA and $\angle PCO = \angle QCA$. Find the locus of the point of intersection of the PQ and the line OA.

RMO - 2006, (17/12/06)

1. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$$

2. Find all positive integers n such that the number $n(2^{n-1}) + 1$ is a perfect square.
3. In how many ways can 7 $X's$ be written so that each unit square contains at most one X and no row is empty in the following figure?



4. Let PA be a common chord of circles C_1 and C_2. Extend PA to Q such that A is midpoint of PQ. Let the tangent to the circle C_1 drawn at P intersect C_2 at R and the tangent to the circle C_2 drawn at P intersect C_1 at S. Show that P, Q, R, S are concyclic.
5. Let ΔABC be a triangle with $\angle B$ as an obtuse angle and $\angle A < 60^0$. Let P be a point on the side AB such that $\angle CPB = 60^0$. Let D be the point on CP which also lies on the internal angle bisector of $\angle A$. If $\angle CBD = 30^0$, prove that CP trisects $\angle ACB$.
6. A person starts from the origin $O(0,0)$ in the $X-Y$ plane. He takes steps of one unit along the X-axis (positive as well as negative direction) or the Y-axis (positive as well as negative direction). Travelling in this manner, find the total number of ways he can reach $A(4,3)$ by using exactly 11 steps?

7. Find all real numbers x, y, z such that
$\frac{1}{xy} = \frac{x}{z} + 1, \quad \frac{1}{yz} = \frac{y}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1.$
8. Find all the positive integers (x, y, z) such that
$xyz = 5(x + y + z)$.

RMO - 2007

1 Let ΔABC be an acute angled triangle; AD be the bisector of $\angle BAC$ and D on BC; and BE be the altitude from B on AC.
Show that $\angle CED > 45°$.

2 Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c - a, c - b) = 1$. Suppose there exists an integer d such that $a + d, b + d, c + d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c + d = l^2 + m^2$.

3 Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

4 How many 6-digit numbers are there such that:
(a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$ and
(b) any digit that appears in the number appears at least twice?

5 A trapezuim $ABCD$, in which $AB \parallel CD$, is inscribed in a circle with center O. Suppose the diagonals AC and BD of the trapezium intersect at M, and $OM = 2$.
(a) If $\angle AMB$ is $60°$, find with proof, the difference between the lengths of the parallel sides.
(b) If $\angle AMD$ is $60°$, find with proof, the difference between the lengths of the parallel sides.

6 Prove that
(a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$;
(b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$;
(c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

RMO - 2008

1 On a semicircle with diameter AB and centre S, points C and D are given such that point C belongs to arc AD. Suppose $\angle CSD = 120°$. Let E be the point of intersection of the straight lines AC and BD and F is the point of intersection of straight lines AD and BC.
Prove that $EF = \sqrt{3}AB$.
2 Solve the system of equations

$$\begin{aligned} x + y + z &= 2; \\ (x+y)(y+z) + (y+z)(z+x) + (z+x)(x+y) &= 1; \\ x^2(y+z) + y^2(z+x) + z^2(x+y) &= -6. \end{aligned}$$

3 Prove that for every positive integer n and a non-negative real number a the following inequality holds:
$n(n+1)a + 2n \geq 4\sqrt{a}(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n})$.

4 Determine all the natural numbers n such that 21 divides $2^{2^n} + 2^n + 1$.

5 Let N be a ten digit positive integer divisible by 7. Suppose the first and the last digit of N are interchanged and the resulting number (not necessarily ten digit) is also divisible by 7 then we say that N is a good integer. How many ten digit good numbers are there?

6 Let $BCDK$ be a convex quadrilateral such that $BC = BK$ and $DC = DK$. A and E are points such that $ABCDE$ is a convex pentagon such that $AB = BC$ and $DE = DC$ and K lies in the interior of the pentagon $ABCDE$. If $\angle ABC = 120°$ and $\angle CDE = 60°$ and $BD = 2$ then determine area of the pentagon $ABCDE$.

RMO-2009

1. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. Find $\angle BAC$.

2. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.

3. Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than 2009^{182}.

4. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

5. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.

(i) Prove that the distance between any two points on the boundary of Γ does not exceed 1.

(ii) If X and Y are two distinct points inside Γ, prove that there exists a point Z on the boundary of Γ such that $XZ + YZ \leq 1$.

6. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

CRMO ans RMO 2010

1. Let $ABCDEF$ be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of tringale OAF is the geometric mean of those of OAB and OEF; and the area of triangle OBC is the geometric mean of those of OAB and OCD. Prove that the area of triangle OED is the gemetric mean of those of OCD and OEF.

2. Let $P_1(x) = ax^2 - bx - c, P_2(x) = bx^2 - cx - a, P_3(x) = cx^2 - ax - b$ be three quadractic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

3. Find the number of 4-digit number (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

5. Let ABC be a traingle in which $\angle A = 60^o$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.

6 For each integer $n \geq 1$, define $a_n = \left[\frac{n}{[\sqrt{n}]}\right]$, where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \cdots, 2010\}$ for which $a_n \geq a_{n+1}$.

CRMO and RMO 2011

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K. Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$.

2. Let $(a_1, a_2, a_3 \cdots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers $1, 2, 3 \cdots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.

3. A natural number n is chosen strictly between two consecutive perfect aquares. The smaller of these two squares is obtained by subtracting k form n and the larger one is obtained by adding l to n. Prove that $n - kl$ is a perfect square.

4. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \cdots, A_20$ in that order. Find the number of ways in which three isdes of K can be chosenso that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)

5. Let ABC be a triangle and let BB_1, CC_1 be respectivly the bisectors of $\angle B, \angle C$ with B_1 on AC and C_1 on AB. Let $E, F,$ be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that $AD = EF$.

6. Find all pairs (x, y) of real numbers such that
$16^{x^2+y} + 16^{x+y^2} = 1$.

RMO 2012

1. Let $ABCD$ be a convex quadrilateral such that $\angle ADC = \angle BCD > 90°$. Let E be the point in which the line AC intersects the line parallel to AD through B and let F be the point in which the line BD intersects the line parallel to BC through A. Prove that EF is parallel to CD.

2. Let $P(x) = x^2 + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all roots are real, find the remaning coefficients.
(Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.)

3. Find all natural numbers x, y, z such that
$(2^x - 1)(2^y - 1) = 2^{2^z} + 1$.

4. Let a, b, c be positive real numbers such that $abc(a + b + c) = 3$. Prove that $(a + b)(b + c)(c + a) \geq 8$,

and determine when equality holds.

5. Let AL and BK be the angle bisectors in the non-isosceles triangle ABC, where L lies on BC and K lies on AC. The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.

6. A computer program generated 175 positive integers at random, none of which had a prime divisor greater than 10. Prove that there are three numbers amongest them whose product is cube of an integer.

RMO 2013

1. Find all positive integers m such that $(m - 1)!$ is divisible by m.

2. H is the orthocentre of triangle ABC, D is any point on BC. If a circle described with centre D and radius DH meets AH produced in E, prove that E lies on the circumecentre of $\triangle ABC$.

3. Suppose ABC is an acute-angled triangel with $AB < AC$. Let M be the mid-point of BC. Suppose P is a point on side AB such that, if PC intesects the median AM at E, then $AP = PE$. Prove that $AB = CE$.

4. Let x and y be real numbers such that $x^2 + y^2 = 1$. Prove that
$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy} \geq \frac{3}{1+(\frac{x+y}{2})^2}$.
When does equality occur?

5. Let a_n be the number of sequences of n terms formed using only the digits $0, 1, 2,$and 3 in which 0 occurs an odd number of times. Find a_n.

6. Find all positive integers n such that the product of all the positive divisors of n is equal to n^3.